Shortcomings of Probability Density and Current density removed by Dirac :- Pg-510

Let us check whether the Dirac eqn leads to the correct probability density.

The Dirac eqn for a free particle is

$$[E - c\vec{\alpha}.p - \beta mc^2]\psi = 0$$

where E and p are operators given by

$$E \rightarrow i\hbar\frac{\partial}{\partial t}$$

$$p \longrightarrow -i\hbar\nabla$$

So, $i\hbar\frac{\partial\psi}{\partial t} + i\hbar c\vec{\alpha}.\nabla\psi - \beta mc^2\psi = 0$ —(1)

A Hermitian conjugate eqn give (Hermitian conjugate is denoted by dagger †)

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t} - i\hbar c\nabla\psi^\dagger.\vec{\alpha} - \psi^\dagger\beta mc^2 = 0$$

Recall $\vec{\alpha}$ and β are Hermitian.
Multiply ① on the left by $\psi^\dagger$ and ② on the right side by ψ

$$i\hbar \psi^\dagger \frac{\partial \psi}{\partial t} + i\hbar c \psi^\dagger \vec{\alpha} \cdot \nabla \psi - mc^2 \psi^\dagger \beta \psi = 0$$

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} \psi - i\hbar c \nabla \psi^\dagger \cdot \vec{\alpha} \psi - mc^2 \psi^\dagger \beta \psi = 0$$

On subtracting.

$$i\hbar \frac{\partial}{\partial t}(\psi^\dagger \psi) + i\hbar c \nabla \cdot (\psi^\dagger \vec{\alpha} \psi) = 0 \quad ③$$

Eqⁿ of Continuity $\quad \nabla \cdot S(r,t) + \frac{\partial P}{\partial t}(r,t) = 0$

We can thus identify the probability density and current density

$$\left. \begin{aligned} P(r,t) &= \psi^\dagger \psi \\ S(r,t) &= c \psi^\dagger \alpha \psi \end{aligned} \right\} \quad ④$$

The probability density is familiar. The current density expression looks more plausible if we note $c\vec{\alpha}$ is the velocity of the particle in the usual sense

$$i\hbar \frac{dx}{dt} = [x, H] = [x, c\vec{\alpha} \cdot p + \beta mc^2] = i\hbar c \vec{\alpha}$$

$$\frac{dx}{dt} = c\vec{\alpha}$$

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